7.2: Transformations of Initial Value Problems

Theorem 1. (Transforms of Derivatives)

Suppose that the function $f(t)$ is continuous and piecewise smooth for $t \geq 0$ and is of exponential order as $t \to \infty$. Then $\mathcal{L}\{f'(t)\}$ exists (for $s > c$) and

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0).$$

Exercise 1. Find a similar formula for $\mathcal{L}\{f''(t)\}$ and then try to generalize to a formula for $\mathcal{L}\{f^{(n)}(t)\}$.

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f')'(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$
$$= s(s\mathcal{L}\{f(t)\} - f(0)) - f'(0)$$
$$= s^2F(s) - sf(0) - f'(0).$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Example 1. Solve the initial value problem (using Laplace transforms)

$$x'' - x' - 6x = 0; \quad x(0) = 2, \quad x'(0) = -1.$$

$$\mathcal{L}\{x'' - x' - 6x = 0\} \Rightarrow \mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 6\mathcal{L}\{x\} = \mathcal{L}\{0\}$$

$$\Rightarrow [s^2X(s) - sx(0) - x'(0)] - [sX(s) - x(0)] - 6X(s) = 0$$

$$\Rightarrow (s^2 - s - 6)X(s) - 2s + 3 = 0$$

$$\Rightarrow X(s) = \frac{2s-3}{s^2-s-6} = \frac{3/5}{s-3} + \frac{7/5}{s+2}.$$

Thus $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{3/5}{s-3} + \frac{7/5}{s+2}\right\} = \frac{3}{5}e^{3t} + \frac{7}{5}e^{2t}.$

Example 2. Suppose we wish to study the motion of a mass-and-spring system with external force which gives a differential equation

$$x'' + 4x = \sin 3t; \quad x(0) = x'(0) = 0.$$

Solve this equation using Laplace transforms.

$$\mathcal{L}\{x''+4x\} = \mathcal{L}\{\sin 3t\}$$

$$[s^2X(s) - sx'(0) - x(0)] + \cancel{4[sX(s) - x(0)]} = \frac{3}{s^2+9}$$

$$4X(s)$$

$$(s^2+4)X(s) = \frac{3}{s^2+9} \Rightarrow X(s) = \frac{3}{(s^2+9)(s^2+4)} = \frac{3/5}{s^2+4} - \frac{3/5}{s^2+9}.$$

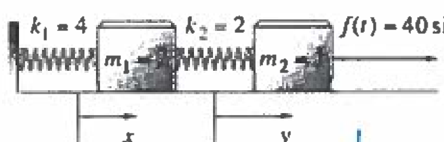
Thus $x(t) = \mathcal{L}^{-1}\{X(s)\}$

$= \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3$

Example 3. In Chapter 4, we will consider systems of differential equations. Here is a glimpse at the power of Laplace transforms. Solve the system

$$\text{(i)} \quad 2x'' = -6x + 2y$$
$$\text{(ii)} \quad y'' = 2x - 2y + 40\sin 3t$$

with initial conditions $x(0) = x'(0) = y(0) = y'(0) = 0$. This is an example of a mass-and-spring system as below.



(i) $\mathcal{L}\{2x''\} = \mathcal{L}\{-6x + 2y\}$

$$2s^2X(s) = -6X(s) + 2Y(s)$$

(ii) $\mathcal{L}\{y''\} = \mathcal{L}\{2x - 2y + 40\sin 3t\}$

$$s^2Y(s) = 2X(s) - 2Y(s) + \frac{120}{s^2+9}$$

So $(s^2+3)X(s) - Y(s) = 0$

$$-2X(s) + (s^2+2)Y(s) = \frac{120}{s^2+9}$$

$$\det\begin{vmatrix} s^2+3 & -1 \\ -2 & s^2+2 \end{vmatrix} = (s^2+1)(s^2+4)$$

Therefore

$$X(s) = \frac{120}{(s^2+1)(s^2+4)(s^2+9)} = \frac{5}{s^2+1} - \frac{8}{s^2+4} + \frac{3}{s^2+9}$$

$$Y(s) = \frac{120(s^2+3)}{(s^2+1)(s^2+4)(s^2+9)} = \frac{10}{s^2+1} + \frac{8}{s^2+4} - \frac{18}{s^2+9}.$$

Thus

$$x(t) = 5\sin t - 4\sin 2t + \sin 3t$$

$$y(t) = 10\sin t + 4\sin 2t - 6\sin 3t.$$

Example 4. Find $\mathcal{L}\{te^{at}\}$.

If $f(t) = te^{at}$ then $f'(t) = e^{at} + ate^{at}$.

Thus $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = s\,\mathcal{L}\{f(t)\}$.

$\quad \| $

$\mathcal{L}\{e^{at}\} + a\,\mathcal{L}\{f(t)\}$.

Thus. $\mathcal{L}\{f(t)\} = \dfrac{\mathcal{L}\{e^{at}\}}{s-a} = \dfrac{1}{(s-a)^2}$.

Exercise 2. Find $\mathcal{L}\{t \sin kt\}$ using the same method as Example 4.

$f(t) = t\sin kt$ then $f(0) = 0$ and $f'(t) = \sin kt + kt\cos kt$, $f'(0) = 0$

and $f''(t) = \cancel{k\cos kt}\ 2k\cos kt - k^2 f(t)$.

Thus $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\}$.

$\quad \|$

$2k\,\mathcal{L}\{\cos kt\} - k^2\mathcal{L}\{f(t)\}$

herefore

$\mathcal{L}\{f(t)\} = \dfrac{2k\,\mathcal{L}\{\cos kt\}}{s^2+k^2} = \dfrac{2ks}{(s^2+k^2)^2}$.

Consider the alternative

$\mathcal{L}\{t\sin kt\} = \int_0^\infty e^{-st}\, t\sin kt\, dt$.

Ain't Nobody got time for that!

Theorem 2. (Transforms of Integrals)

If $f(t)$ is a piecewise continuous function for $t \geq 0$ and is of exponential order, then

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \tfrac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\{\frac{F(s)}{s}\} = \int_0^t f(\tau)d\tau.$$

Example 5. Find $\mathcal{L}^{-1}\{\frac{1}{s^2(s-a)}\}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}(\tau)\,d\tau$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}(\tau)\,d\tau = \int_0^t e^{a\tau}\,d\tau = \frac{1}{a}\left(e^{at}-1\right)$$

$$\int_0^t \frac{1}{a}\left(e^{a\tau}-1\right)d\tau = \frac{1}{a^2}\left(e^{at}-at-1\right).$$

Exercise 3. Find $\mathcal{L}^{-1}\{\frac{3}{s(s+5)}\}$.

$$\mathcal{L}^{-1}\left\{\frac{3}{s(s+5)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{3}{s+5}\right\}d\tau$$

$$= \int_0^t 3e^{-5t}\,d\tau = -\frac{3}{5}e^{-5t}\Big|_0^t = \frac{3}{5}\left(1-e^{-5t}\right).$$

Homework. 1-5, 11-23 (odd) 27-33 (all)